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AÑALYSIS OF A CAUSTIC BY RAY THEORY.(U)

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Analysis of a Caustic by Ray Theory

1. The Problem

A ray may be defined as the locus of propagation of an element of wavefront. The definition includes the plane-wave idealization that the wavefront propagates along its instantaneous normal; a propagating wavefront is discretized by a selection of rays; each ray is bent by refraction in a medium having sound-speed gradient. By definition, rays are orthogonal to wavefronts.

We denote the specific wavefront length, in a vertical plane, by L. Along a ray as the wavefront diverges L increases; as the wavefront converges L decreases and may pass through zero. Passing through zero may indicate a folding or focusing of the wavefront. A caustic is a ray-to-ray locus of an $L = 0$ occurance. A caustic generally indicates a folding of the wavefront. The end points of a caustic may be indicative of focusing.

We specialize to the case of sound-speed varying with depth, z, only, and a horizontal caustic at depth z_*. This imposes horizontal uniformity, without caustic end points.

For the analysis we assume the sound-speed profile

$$c' = \mu z \tag{1}$$

where μ is the gradient at caustic depth, and the zero depth level is defined by the speed, c_*, at the caustic depth. We shall have to decide later as to the extent of the region of analysis applicability about the caustic.

The associated, consistent, ray, wavefront and caustic configuration is depicted in Fig. 1.

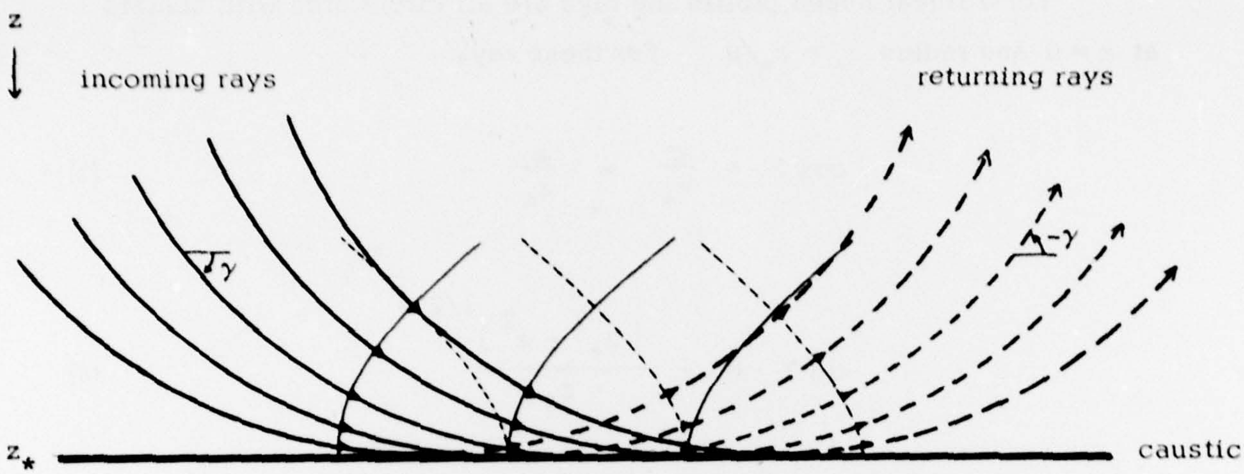

Fig. 1 The caustic, with rays and wavefronts

According to ray theory the spreading factor, amplitude A, for sound intensity is given by

$$A = |L|^{-1} \tag{2}$$

Sign changes in L along a ray are indicative of wavefront foldings.

At a caustic $L = 0$ and $A \to \infty$. Does this mean that every point on a caustic is a focal point? Generally, effectively not. We have both incoming rays and their refracted return (except at caustic ends) and it is the resultant that manifests. We here attempt to determine the resultant by analysis consistent with ray theory. That is, we wish to extend ray tracing for applicability at caustics.

2. The Analysis

For a linear speed profile the rays are all circle arcs with centers at $z = 0$ and radius $z_* = c_*/\mu$. For these rays

$$\cos\gamma = \frac{c}{c_*} = \frac{z}{z_*} \tag{3}$$

$$\sin\gamma = \pm \frac{[z_*^2 - z^2]^{1/2}}{z_*} \tag{4}$$

We here adopt the convention that, should there be a sign duality, the upper sign applies to incoming rays/wavefronts and the lower sign to returning rays/wavefronts. The rays may be expressed by

$$x - x_* = -z_* \sin\gamma \tag{5}$$

where x is introduced as the horizontal linear coordinate. Each ray is identified by x_*; the circle arc center is at x_* and $z = 0$, and the ray touches the caustic at x_*, z_*.

The specific wavefront length may be expressed by

$$L = \delta x_* \sin\gamma \tag{6}$$

And, by ray theory, the normalized wave amplitude may be expressed by

$$A = \pm \frac{1}{\sin\gamma} \tag{7}$$

The wavefronts are obtained by integration of

$$\frac{dx}{dz} = -\tan\gamma \tag{8}$$

which derives from their being orthogonal to the rays. The wavefronts may be expressed by

$$z_* \sin\gamma \mp z_* \ln \frac{1 \pm \sin\gamma}{\cos\gamma} + x - x_* = 0 \tag{9}$$

Each wavefront (line of constant phase) is identified by x_*, where the wavefront touches the caustic.

At the caustic limit the wavefronts are moving horizontally with speed c_*. A steady state configuration is realized from a frame of reference moving with that velocity. We adopt this frame of reference.

At this point we introduce the waveform

$$\varphi = A \cos kx_* \tag{10}$$

which corresponds to the frequency

$$\omega = kc_* / 2\pi \tag{11}$$

By substituting for x_* from Eq. (9) and for A from Eq. (7) we may express the incoming wave by

$$\varphi_i = \frac{1}{\sin\gamma} \cos k\,[x - \theta_i(z)] \tag{12}$$

where

$$\theta_i(z) = -z_* \sin\gamma + z_* \ln \frac{1+\sin\gamma}{\cos\gamma} \tag{13}$$

And the returning wave by

$$\varphi_r = \frac{1}{\sin\gamma} \cos k\,[x - \theta_r(z)] \tag{14}$$

where

$$\theta_r(z) = -z_* \sin\gamma - z_* \ln \frac{1-\sin\gamma}{\cos\gamma} \tag{15}$$

For the resultant, Φ, we add the two fields:

$$\Phi = \varphi_i + \varphi_r \tag{16}$$

In expressing the resultant we use positive γ and $\sin\gamma$ only. Thus for the returning wave the signs of $\sin\gamma$ are reversed in performing the addition:

$$\Phi = \frac{1}{\sin\gamma} \cos k\,[x-\theta] - \frac{1}{\sin\gamma} \cos k\,[x+\theta]$$

$$= \frac{2}{\sin\gamma} \sin k\theta \;\sin kx \tag{17}$$

where

$$\theta \equiv z_* \left[\ln \frac{1+\sin\gamma}{\cos\gamma} - \sin\gamma\right] \tag{18}$$

Let us examine the resultant in the region of the caustic, where $\gamma << 1$. Expansion reveals that θ may be approximated by

$$\theta \approx z_\star \frac{1}{3} \gamma^3 \tag{19}$$

For the supplemental condition that $k\theta << 1$, the resultant may be approximated by

$$\Phi \approx \frac{4\pi}{3} \frac{\omega}{\mu} \gamma^2 \sin kx \tag{20}$$

where we have replaced $kz_\star$ according to Eqs. (1) and (11). From Eq. (4) we derive for small γ, small $\delta z \equiv z_\star - z$, that

$$\gamma^2 \approx \frac{2\mu}{c_\star} \delta z \tag{21}$$

Equations (20) and (21) combine to reveal that the resultant approaches a first-order zero at the caustic:

$$\Phi \approx \frac{8\pi}{3} \frac{\omega}{c_\star} \delta z \sin kx \tag{22}$$

By ray theory the spreading factor, amplitude A, is zero at a caustic. This analysis, however, does not apply in the region of caustic end points.

It is relevant to determine the depth and value of maximum amplitude for the resultant, Φ, provided we can show that it occurs near the caustic in the region of analysis applicability. We denote the level

by z_m and values there by subscript m. We first assume that $\gamma << 1$ and later show this to be so.

For small γ we approximate θ by Eq. (19) and express the amplitude factor of Eq. (17) by

$$A = \frac{2}{\gamma} \sin\left[\frac{2\pi}{3} \frac{\omega}{\mu} \gamma^3\right] \tag{23}$$

The extremal occurs where $dA/d\gamma = 0$. This yields

$$\tan\left[\frac{2\pi}{3} \frac{\omega}{\mu} \gamma_m^3\right] = 3\left[\frac{2\pi}{3} \frac{\omega}{\mu} \gamma_m^3\right] \tag{24}$$

$$\frac{2\pi}{3} \frac{\omega}{\mu} \gamma_m^3 \approx 1.32 \tag{25}$$

$$\gamma_m \approx \left[1.32 \frac{3}{2\pi} \frac{\mu}{\omega}\right]^{1/3} \approx 0.86 \left[\frac{\mu}{\omega}\right]^{1/3} \tag{26}$$

$$A_m = \frac{1.94}{\gamma_m} \approx 2.26 \left[\frac{\omega}{\mu}\right]^{1/3} \tag{27}$$

In order to evaluate γ_m we must specify μ/ω in Eq. (26). Even for the relatively high value of 10^{-3} we obtain $\gamma_m \approx 0.086$ which satisfies our condition of smallness; corresponding thereto $A_m \approx 22.6$, a considerable amplification.

Equation (25) expresses the phase angle $k\theta_m$ which appears in φ_i and φ_r as combined for Eq. (17). We note that z_m occurs somewhat below $k\theta = \pi/2$ which is the level where φ_i, φ_r, and Φ are all in phase.

From Eq. (3) we obtain

$$z_m = \frac{c_*}{\mu} \cos \gamma_m \tag{28}$$

For $c_* = 1500 \text{ m sec}^{-1}$, $\mu = 10^{-1} \text{ sec}^{-1}$, and $\omega = 100$ Hz giving $\gamma_m = 0.086$, we obtain

$$z_* - z_m \approx 54 \text{ m} \tag{29}$$

Analysis applicability requires that the speed profile be linear about the caustic. This linearity must extend to beyond the z_m level for the maximum-amplitude analysis to be valid.

3. The Result

The payoff of the analysis lies in Eq. (27). We can split the resultant amplitude, Φ, between φ_i and φ_r to get at an effective lower bound on L in computing the amplitude factor along a ray. The analysis suggests the approximate lower bound:

$$L_m \approx \omega^{-1/3} \mu^{1/3} \tag{30}$$

However we must not forget the normalizing factor. Equation (7) indicates how we have normalized L. We now write

$$L = \sin(\gamma - \gamma_0) \tag{31}$$

where γ_0 is the ray angle at caustic tangency; that is, we do not require the caustic to be strictly horizontal. The missing factor in L of Eq. (31), lost by normalizing, can be obtained from L, along the ray, before and/or after the ray passes through the caustic. Adding this factor to Eq. (30) yields

$$L_m \approx \frac{L_c}{\gamma_c - \gamma_0} \omega^{-1/3} \mu^{1/3} \tag{32}$$

This is where ray theory has led us. The approximation omits caustic curvature and lateral variability in rays. Equation (32) also does not apply near caustic end (branch) points where interference is partial or nil and strong focusing may occur. It remains to reconcile the results with full wave theory and energy considerations.